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Complex Oscillation Theory in Some Complex Domains

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複素領域における振動問題について

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ABSTRACT

We treat linear homogeneous differential equations in the complex plane with entire coefficients. We are concerned with the complex oscillation to describe the distributions of zeros of entire solutions. The case with exponential polynomials are mainly considered, in particular, $f'' + (e^{P_1(z)} + e^{P_2(z)} + q(z))f = 0$ is investigated. We give an survey on the research of this equation, and construct examples for exceptional cases.

要　旨

複素平面上で整関数を係数とする線形同次微分方程式を取り扱う。与えられた方程式の整関数解の零点を記述する複素振動について考える。特に，係数が指数多項式である２階の方程式 $f'' + (e^{P_1(z)} + e^{P_2(z)} + q(z))f = 0$ の複素振動を調べることに問題意識をおく。この方程式の先行研究についての解説を与えると共に，除外的な場合の例を構成する。

1　Introduction

In the complex plane, we consider entire solutions of linear differential equation

$$f'' + A(z)f = 0, \tag{1}$$

where $A(z)$ is an entire function.

Let $f(z)$ be an entire function. We use the standard notations of the value distribution theory due to Nevanlinna, see e.g., [4], [5], [7] and [10]. We denote by $\sigma(f)$ the growth order of $f(z)$, and denote by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}$$

the exponent of convergence of the zero-sequence of $f(z)$. By means of Nevanlinna theory, if $A(z)$ is a transcendental entire function the non-trivial solutions are of infinite order.

We are concerned with the problem under what conditions solutions of (1) have many zeros, or some solution does not have many zeros. The research in this direction is called complex oscillation theory, see e.g., [1], [10], [12].

To investigate the distribution of zeros of entire solutions of (1), we consider their behaviors on rays (half lines) and in sectors. Write a ray $L_\theta = \{re^{i\theta} \in \mathbb{C} \mid 0 \leqq r < \infty\}$ and a sector

$$S(r, a, b) = \{z \mid |z| > r, a < \arg z < b\}.$$

Let $\alpha, \beta \in \mathbb{C}$ and $n \in \mathbb{N}$. For a polynomial

$$P(z) = (\alpha + i\beta)z^n + \cdots + a_0,$$

we define for each θ

$$\delta(P, \theta) = \alpha\cos\theta - \beta\sin\theta.$$

One of the method to show that the solution $f(z)$ of (1) has many zeros is the following, see, e.g., [2], [3], [9]. First we assume that $f(z)$ has few zeros. Using this assumption and the lemmas in value distribution theory, e.g., the estimates of logarithmic derivatives [6], an auxiliary function behaves small in growth on

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some rays. By means of Phragmén–Lindelöf type theorem, see e.g., [11], the auxiliary function behaves small in growth on some sectors or whole complex plane, which yields a contradition.

2 Linear differential equations with an exponential polynomial coefficients

We recall the result due to Bank, Laine and Langley [3]

Theorem A. Suppose that $k \geq 2$, that $P(z)$ is a polynomial of degree $n > 0$, that $R(z)$ is a rational function not vanishing identically, that Π_1 is an entire function of order $\sigma(\Pi_1) < n$, and that β is a constant with $0 \leq \beta < 1$. Suppose further that either β is irrational or $0 < \beta < 1 - \frac{1}{k}$ or $\beta = 0$ and $\Pi_1 \equiv 0$. Suppose finally that $A(z)$ is an entire function of finite order such that for some real θ_0 with $\delta(P, \theta_0) = 0$ and some $\alpha > 0, r_0 > 0$, we have

$$\left| A(z) - R(z)e^{P(z)} - \Pi_1(z)e^{\beta P(z)} \right| \leq |z|^{\lambda}$$

in the sector $S(r_0, \theta_0 - \alpha, \theta_0 + \alpha)$, where $k + \lambda < nk$ if $n > 1$ and $\lambda < -k$ if $n = 1$. Then all non-trivial solution f of

$$y^{(k)} + A(z)y = 0$$

satisfy $\lambda(f) = \infty$.

We consider the case $k = 2$ in Theorem A. If $\beta = 2$, then the assumption $\beta < 1 - \frac{1}{k}$ of Theorem A is not satisfied. As the authors of [3] pointed out, there is an example which shows that $\lambda(f) = \infty$ does not hold. In fact, the function $e^{e^{P(z)}}$ satisfies

$$f'' - \left((P')^2 e^{2P} + (P'' + (P')^2)e^P\right) f = 0,$$

and $e^{e^{P(z)}}$ has no zeros.

Theorem B. Suppose that $P(z)$ and $Q(z)$ are non-constant polynomials such that $\deg Q \leq \deg P$ and if $\deg P = \deg Q = n$, then

$$P(z) = a_n z^n + \cdots + a_0, \quad Q(z) = b_n z^n + \cdots + b_0$$

are such that b_n/a_n is non-real. Let $T(z)$ be a polynomial such that $\deg T + k < k \deg P$ and that $T(z)$ vanish identically, if $\deg P = 1$. If $k \geq 2$, and if $R(z)$, $S(z)$ are polynomials, $R(z)$ not vanishing identically, then all non-trivial solutions $f(z)$ of

$$(2) \qquad y^{(k)} + (R(z)e^{P(z)} + S(z)e^{Q(z)} + T(z))y = 0$$

satisfy $\lambda(f) = \infty$.

It is also pointed out in [3] that the case $\deg P = \deg Q$ and $b_n/a_n > 0$ seems difficult to treat, for example

$$(3) \qquad y'' + (e^{4z} + \lambda e^{3z})y = 0,$$

where λ is a constant.

3 The $\frac{1}{16}$-theorem

In this section we consider the case $A(z) = e^{P(x)} + Q(z)$, where $P(z)$ is a polynomial of degree p, and $Q(z)$ is an entire function of order less that p. In [2], the case $A(z) = e^z - K$, where K is a complex constant is discussed. For $K = \frac{1}{16}$,

$$f'' + (e^z - K)f = 0$$

possesses two linearly independent solutions $f_1(z)$ and $f_2(z)$ such that max $(\lambda(f_1), \lambda(f_2)) = 0$. For all other constant K we have

$$\max(\lambda(f_1), \lambda(f_2)) \geq 1.$$

Further, it is proved that there exist two linearly independent solutions $f_1(z)$ and $f_2(z)$ satisfying max $(\lambda(f_1), \lambda(f_2)) = 1$ for all $K = \frac{q^2}{16}$, with $q \geq 3$ odd integer.

Theorem C. Suppose that

$$(4) \qquad f'' + (e^{P(z)} + Q(z))f = 0$$

admits a non-trivial solution $f(z)$ such that $\lambda(f) < p$. Then $f(z)$ has no zeros, $Q(z)$ is a polynomial and

$$Q(z) = -\frac{1}{16}(P'(z))^2 + \frac{1}{4}P''.$$

Moreover, (4) admits in this case two linearly independent zero-free solutions.

4 The two terms case

In this section, we are concerned with the second order case in (2), in which we relax the condition on $T(z)$. In (5) below, we allow that $q(z)$ could be transcendental.

The authors consider the following equaton, in [9].

$$(5) \qquad f'' + (e^{P_1(z)} + e^{P_2(z)} + q(z))f = 0,$$

where $q(z)$ is an entire function and $P_j(z)$, $j = 1, 2$ are non-constant polynomials

$$(6) \qquad P_1(z) = \zeta_1 z^n + \ldots,$$

$$(7) \qquad P_2(z) = \zeta_2 z^m + \ldots,$$

with $\zeta_j \neq 0$, $j = 1, 2$. Concerning the order condition, it is assumed that $\rho(q) < \max(n, m)$. In case $q(z)$ is a polynomial, (5) is included in (2). Below we suppose that $q(z)$ is transcendental. It is showed the following

Theorem D.

(i) If $n \neq m$, then $\lambda(f) = \infty$ for any non-trivial solution $f(z)$ of (5).

(ii) If $n = m$ and $\zeta_1 = \zeta_2$, then $\lambda(f) \geq n$ for any non-trivial solution $f(z)$ of (5).

(iii) Suppose that $n = m$ and $\zeta_1 \neq \zeta_2$. If ζ_1/ζ_2 is non-real, then $\lambda(f) = \infty$ for any non-trivial solution $f(z)$ of (5).

Further we obtained the following result [8]

Theorem E Consider the equation (5) when $n = m$ and $\rho > 0$.

(i) If $0 < \rho < 1/2$, then for any nontrivial solution of (5) we have $\lambda(f) \geq n$.

(ii) Suppose that $q(z) \equiv 0$ in (5). If $3/4 < \rho < 1$, then for any non-trivial solution of (5) we have $\lambda(f) \geq n$.

Suppose that $n = m$ in (5), (6) and (7), and ρ is real positive. As we mentioned above, the cases $\rho = 1/2$ and $\rho = 3/4$ are the exceptional cases.

5 Examples

We construct examples for the case $\rho = 1/2$ and $\rho = 3/4$. We remark that $e^{P_1(z)} + e^{P_2(z)} + q(z)$ can be written

$$e^{P_1(z)} + e^{P_2(z)} + q(z) = H_0(z) + H_1(z)e^{\zeta_1 z^n} + H_2(z)e^{\zeta_2 z^n},$$

where $H_j(z) \not\equiv 0$, $j = 0, 2, 3$ are exponential polynomials of order less than n or general polynomials. Thus we write (5) as

$$f'' + (H_0(z) + H_1(z)e^{\zeta_1 z^n} + H_2(z)e^{\zeta_2 z^n})f = 0. \tag{8}$$

Let a, b, c and s be complex numbers. We set

$$h(z) = ae^z + be^{sz} + cz \tag{9}$$

and

$$f(z) = e^{h(z)}. \tag{10}$$

Then we have

$$\frac{f''(z)}{f(z)} = a^2e^{2z} + b^2s^2e^{2sz} + a(2c+1)e^z + bs(2c+s)e^{sz} + 2abse^{(1+s)z} + c^2$$

If $a = 0$ or $b = 0$, then $f(z)$ satisfies an equation of the form (8).

Other possibilities when $f(z)$ given in (10) satisfies an equation of the form (8), we consider the cases $s = 0, s = \frac{1}{2}$ and $s = 1$ below.

(i) Set $s = 0$. Then we see that

$$f'' - (a^2e^{2z} + a(2c+1)e^z + c^2)f = 0$$

possesses a zero free solution

$$f(z) = e^{ae^z + cz + b}, \tag{11}$$

which corresponds to the case $\rho = \frac{1}{2}$.

(ii) Set $s = \frac{1}{2}$. Then we see that

$$f'' - \left(a^2e^{2z} + abe^{\frac{3}{2}z} + \frac{1}{4}b(4c+1)e^{\frac{z}{2}} + \frac{1}{4}(4a + b^2 + 8ac)e^z + c^2\right)f = 0 \tag{12}$$

possesses a zero free solution

$$f(z) = e^{ae^z + be^{\frac{1}{2}z} + cz}. \tag{13}$$

Further, setting $c = -\frac{1}{4}$ in (12) and (13), we obtain that

$$f'' - \left(a^2e^{2z} + abe^{\frac{3}{2}z} + \frac{1}{4}(2a + b^2)e^z + \frac{1}{16}\right)f = 0 \tag{14}$$

has a solution

$$f(z) = e^{ae^z + be^{\frac{1}{2}z} - \frac{1}{4}z}. \tag{15}$$

Moreover, we set $a = -\frac{b^2}{2}$ in (14) and (15), which implies that

$$f'' - \left(\frac{1}{4}b^4e^{2z} - \frac{1}{2}b^3e^{\frac{3}{2}z} + \frac{1}{16}\right)f = 0$$

possesses a zero free solution

$$f(z) = e^{-\frac{1}{2}b^2e^z + be^{\frac{1}{2}z} - \frac{1}{4}z},$$

which corresponds to the case $\rho = \frac{3}{4}$.

(iii) Set $s = 1$. Then we see that

$$f'' - ((a+b)^2e^{2z} + (a+b)(2c+1)e^z + c^2)f = 0$$

possesses a zero free solution

$$f(z) = e^{(a+b)e^z + cz},$$

which corresponds to the case $\rho = \frac{1}{2}$.

6 Remarks

We assume that $H_j(z) \not\equiv 0, \zeta_j \neq 0, j = 1, 2$ in (8), and set $\rho = \zeta_2/\zeta_1$. Theorem D states that all non-trivial solutions of second order equation (8) have infinitely many zeros if ρ is non real, in which the exponent of convergences of zero sequences of them are ∞.

We suppose that ρ is real, and assume that $0 < \rho < 1$ without loss of generalities. Examples in Section 4 show that there exist non-trivial zero free solutions when $\rho = 1/2$ and $\rho = 3/4$. Theorem E states that any non-trivial solution $f(z)$ of (8) satisfies $\lambda(f) \geq n$ if $0 < \rho < 1/2$. It is also mentioned that when

$q(z) \equiv 0$ any non-trivial solution $f(z)$ of (8) satisfies $\lambda(f) \geq n$ if $3/4 < \rho < 1$.

Below we state open questions in connecting with these results. (i) We should consider the problem whether we can remove the condition $q(z) \equiv 0$ in the second assertion of of Theorem E. (ii) It is a most curious problem what happens when $1/2 < \rho < 3/4$. (iii) We are also interested in the problem whether it is possible to show $\lambda(f) = \infty$ in stead of $\lambda(f) \geq n$ when $0 < \rho < 1/2$ or $3/4 < \rho < 1$.

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